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On the Non-Trace-Valued Forms*

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INTRODUCTION

Let $\xi \to \bar{\xi}$ be an involutory antiautomorphism of the divison ring k and E a k-vector space. A hermitian form Φ on E is said to be *trace-valued* if, for all $x \in E$, $\Phi(x, x)$ is a trace, i.e ., of the form $\xi + \bar{\xi}$ for some $\xi \in k$ [1]. If the characteristic is not 2 or if the center of k is not fixed under the involution then all Φ are trace-valued.

Non-trace-valued forms are known as rather unmanageable—even in finite dimensions: nondegenerate isotropic planes may fail to be hyperbolic, the cancellation theorem is not valid and so Witt's theorem does not hold either. This last fact seems to entail a particularly unfavourable verdict on the non-trace-valued forms as the classic theory of quadratic forms pivots on this theorem (see also the contention supported in [6, p. 249]).

It is the purpose of this paper to demonstrate that matters may be looked at differentently and that the classification problem on subspaces in the non-trace-valued situation may be successfully attacked by what might be called the lattice method. The ground for this method has been laid in [2].

In order to give perspicuousness to the constituent parts we have stripped the problem of all unnecessary complications. In particular, we have made the following assumptions throughout the paper:

$$\xi \to \bar{\xi} \text{ is the identity (and thus } k \text{ commutative),} \tag{1}$$

$$\operatorname{char} k = 2 \qquad \text{and} \qquad [k : k^2] < \infty. \tag{2}$$

Thus forms will be symmetric in what follows and zero is the only trace in k. In order to give prominence to the role played by the lattices we often

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restricted ourselves to perfect fields. Even under these provisos the difficulties which still remain for the classification problem are formidable.

We shall always have $\dim E \leqslant \aleph_0$ and be inerested primarily in the infinite case. However, since precise information is scarce even in the finite dimensional situation we have included at the end a section on such forms; in particular we give there the appropriate version of Witt's theorem for forms over perfect fields. The principal goal of the paper is to show the method for proving Theorem 1.

Notation. We set $E^* = \{x \in E \mid \Phi(x, x)$ is a trace$\}$. E^* is a linear subspace with $\dim E/E^* \leqslant [k : k^2]$. For $X \subset E$ a subspace we set $X^* = X \cap E^*$; equipped with its induced form X^* is alternate. $\|X\| := \{\Phi(x, x) \mid x \in X\}$. By (2), $\dim E/E^*$ will always be finite; the discussion which follows actually includes that of spaces E with $\|E\|$ finite dimensional over k^2 and k an arbitrary field.

1. The Indices

Let E be a vector space and $\mathscr{L}$ the lattice of all linear subspaces. Equip E with a nondegenerate form and let $^\perp$: $\mathscr{L} \to \mathscr{L}$ be the operation of taking the orthogonal. With each subspace $F \subset E$ we can associate the orthostable lattice $\mathscr{V}(F, E^*)$ generated in $\mathscr{L}$ by the elements F and E^*. We require that (0) and E are elements of the lattice.

The particular lattices studied in this paper have the nice feature of being finite. To have an example one may look up Fig. 1 in 6.1.

We call *indices* the dimensions of quotients of neighbouring elements $X, Y \in \mathscr{V}(F, E^*)$; they obviously are invariants of F modulo metric automorphisms of E, i.e., they are invariants of the orbit in $\mathscr{L}$ of F under the orthogonal group of E. Each element of this group leaves E^* invariant and $E^{*\perp}$ pointwise fixed (if $x \in E^{*\perp}$, $y \in E$ then $(Tx - x, y) = (Tx, y) - (Tx, Ty) = (Tx, y - Ty) = 0$ since $y - Ty \in E^*$ so $Tx - x = 0$). Therefore, in order that at least *one* isometry $T_0 : F \to \bar{F}$ between two subspaces $F, \bar{F}$ can be extended to all of E it is necessary that

$$F \cap E^{*\perp} = \bar{F} \cap E^{*\perp}. \tag{3}$$

We shall show that in important cases these obvious invariants constitute a *complete* set of invariants for the orbit of a subspace.

2. The Main Result

In this paper we shall prove the following

THEOREM 1. *Let Φ be a nondegenerate not necessarily alternate bilinear form on the $\aleph_0$-dimensional k-vector space E where* $\operatorname{char} k = 2$ *and*

$[k:k^2]<\infty$. Let $F, \bar{F} \subset E$ be subspaces with (3) such that the indices attached to F by $\mathscr{V}(F, E^*)$ equal the corresponding cardinals of $\bar{F}$. In order that there exists an isometry $T: E \to E$ with $TF = \bar{F}$ each of the following assumptions is sufficient.

(j) $F^\perp = (0)$ *and* $E^* = E^{*\perp\!\!\perp}$,

(jj) $F^\perp \subset E^*$ *and* $[k:k^2]=1$ *and* $E = E^* \oplus E^{*\perp}$,

(jjj) $F \subset E^*$ *and* $[k:k^2]=1$ *and* $E^* = E^{*\perp\!\!\perp}$,

(jv) $F \subset F^\perp$ *and furthermore* $[k:k^2]=1$ *or* $E^* = E^{*\perp\!\!\perp}$,

(v) $F^\perp \subset E^*$ *and* $[k:k^2]=1$ *and* $E^{*\perp}=(0)$ *and* $\dim F/F^* = \dim F^{*\perp\!\!\perp}/F^{*\perp\!\!\perp *} = \dim F^{\perp\!\!\perp}/F^{*\perp\!\!\perp} = 1$, $\dim F^{*\perp}/F^{*\perp *} = 0$, $F^{\perp\!\!\perp} \cap F^{*\perp} = F^{*\perp\!\!\perp} \cap F^\perp$, $(F \cap F^\perp)^{\perp *} = (F \cap F^\perp)^\perp$.

Before turning to he proofs we shall have to delve into some general considerations.

3. What Is the "Lattice Method"?

The problem envisaged in Theorem 1 is to decide for *specific* $F, \bar{F}$ whether there is an isometry of E with $TF = \bar{F}$ or not. The easiest nontrivial situations arise for F dense and E^* closed ((j) of Theorem 1). Let us consider this particular case:

The desired isometry T is constructed recursively. Suppose we wish to start out by defining T on the line (x_0). How should we define Tx_0? Metric requirements are not the only problem. If the construction is not to be doomed at the outset then the vector $\bar{x}_0 = Tx_0$ has to be picked in such a fashion that the line $(\bar{x}_0)$ has the same order theoretic ubiety relative to $\mathscr{V}(\bar{F}, E^*)$ as has the line (x_0) to $\mathscr{V}(F, E^*)$. It turns out that this presents a difficulty when we are in the case

$$x_0 \in (F + E^{*\perp}) \cap E^* \qquad \text{but} \qquad x_0 \notin (F \cap E^*) + (E^{*\perp} \cap E^*). \tag{4}$$

Now, why difficulties with this particular lattice element $S = (F + E^{*\perp}) \cap E^*$ among 37 others? It happens that it is one of the three join irreducible elements that do not generate a prime filter (see Fig. 1). Our difficulty is not due to an unsuitably chosen proof of existence for T. If S does not generate a prime filter in $\mathscr{V}(F, E^*)$ then the choice of Tx_0 *is* overdetermined when x_0 satisfies (4) and additional considerations at this stage of the recursive construction are needed. But then, how should one see in a concrete situation which lattice elements are irreducible and generate nonprime filters? The answer is: work out the lattice $\mathscr{V}(F, E^*)$.

Working out $\mathscr{V}(F, E^*)$ may be difficult (even a posteriori verification of diagrams can be tedious). This is so because $\mathscr{V}(F, E^*)$ gives an *intrinsic* picture of the complexity of the problem. It is indispensable to start out with a detailed investigation of the lattice. This is what we mean by the lattice method here.

An application of this method to *quadratic* forms in the case of characteristic 2 is given in [3].

4. The General Setup of the Proofs when Φ Is Not Trace-Valued and $\mathscr{V}$ Not Distributive

4.1. Let $\tau : A \to \bar{A}$ be a lattice isomorphism $\mathscr{V}(F, E^*) \cong \mathscr{V}(\bar{F}, E^*)$ which sends F into $\bar{F}$ and which respects indices and $\perp$ and $*$. As in [2]—a paper with which the reader is assumed to be familiar—one tries to construct two ascending sequences $W_0 \subset W_1 \subset W_2, \ldots, \bar{W}_0 \subset \bar{W}_1 \subset \bar{W}_2, \ldots,$ of finite dimensional subspaces of E together with a sequence of isometries $T_i : W_i \to \bar{W}_i$ such that T_{i+1} extends T_i and $\bigcup W_i$ and $\bigcup \bar{W}_i$ are all of E and the T_i are "compatible" with the lattices $\mathscr{V}$, $\bar{\mathscr{V}}$, i.e., $T_i(W_i \cap A) = \bar{W}_i \cap \bar{A}$ for all $A \in \mathscr{V}$.

If such sequences exist it is evident that they define an isometry T of E which induces τ. Furthermore it is clear that there are infinitely many ways of selecting the sequences. It is done in such a manner that the following "distributivity" holds for an *arbitrary family* of elements $A_\iota \in \mathscr{V} : \bigcap_\iota (W_i + A_\iota) = W_i + \bigcap_\iota A_\iota$ and, of course, the corresponding property for each $\bar{W}_i$ of the second sequence. This construction amounts to solving the following

4.2. *Construction problem.* There are given finite dimensional subspaces $W, \bar{W} \subset E$ and an isometry $T : W \to \bar{W}$ with (A) $T(W \cap A) = \bar{W} \cap \bar{A}$ for all $A \in \mathscr{V}$ and (B) $\bigcap_\iota (W + A_\iota) = W + \bigcap_\iota A_\iota$ and the corresponding property ($\bar{\text{B}}$) for $\bar{W}$ with respect to $\bar{\mathscr{V}}$. There is furthermore given a vector $x \in E \backslash W$. One then has to (I) choose a finite dimensional subspace $W_1 \supset W \oplus (x)$; (II) construct a subspace $\bar{W}_1$ such tat T extends to an isometry $T_1 : W_1 \to \bar{W}_1$; (III) verify that $W_1, \bar{W}_1, T_1$ satisfy again (A), (B), ($\bar{\text{B}}$).

As shown in [2] the filter $\mathscr{M}(x, W_n) = \{Z \in \mathscr{V}(F, E^*); x \in W_n + Z\}$ is crucial. In order to be able to verify (B) and ($\bar{\text{B}}$) we must be sure that $\mathscr{M}$ is prime in those instances where it is generated by a join irreducible element. This is certainly the case when the lattice is distributive.

4.3. *The strategy.* From the proof of the main theorem in [2] we can extract the following

Principle I. *Assume that* $\dim E/E^* < \infty$ *and that* $\mathscr{V}(F, E^*)$ *is finite and distributive and that there are finite dimensional spaces* $W_0, \bar{W}_0 \subset E$ *and an isometry* $T_0 : W_0 \to \bar{W}_0$ *such that the following hold:*

If $X \neq X^*$ is a join irreducible element of $\mathscr{V}(F, E^*)$ then there exists a subspace $H \subset W_0$ which is a linear supplement in X of the immediate antecedent X_0 of X, (5)

$$T_0 : W_0 \cap A \to \bar{W}_0 \cap \bar{A} \qquad \textit{for all } A \in \mathscr{V}(F, E^*), \tag{6}$$

$$(W_0 + A) \cap (W_0 + B) = W_0 + (A \cap B) \qquad \textit{for all } A, B \in \mathscr{V}(F, E^*), \tag{7}$$

$$(\bar{W}_0 + \bar{A}) \cap (\bar{W}_0 + \bar{B}) = \bar{W}_0 + (\bar{A} \cap \bar{B}) \qquad \textit{for all } \bar{A}, \bar{B} \in \mathscr{V}(\bar{F}, E^*), \tag{8}$$

then T_0 can be extended to an isometry T of E with $TA = A^\tau$ for all $A \in \mathscr{V}(F, E^*)$.

Remark. Just as in [2] the lattice need not be finite and a more general principle (not needed in what follows) can be formulated.

The next question is: what can be done if $\mathscr{V}$ is not distributive? The idea behind all proofs that follow is to arrange the very first step of the recursive construction in such a way that the join irreducible lattice elements D which do not generate prime filters are excluded from the role as generators of the filters $\mathscr{M}(x, W_n)$ associated with the construction problem 4.2. This is achieved by putting into W_0 a supplement of D_0 in D (D_0 the immediate antecedent of D).

We summarize:

PRINCIPLE II. *Assume that* $\dim E/E^*$ *is finite and that the initial triple* $(W_0, \bar{W}_0, T_0)$ *satisfies* (5), (6), (7), (8) *and the following*:

If X is join irreducible and the principle filter which it generates is not prime then there exists a subspace $H \subseteq W_0$ which is a linear supplement in X of the immediate antecedent X_0 of X. (9)

Then T_0 can be extended to an isometry T of E with $TA = A^\tau$.

5. Classification Theorem for $\aleph_0$-Dimensional Spaces

We shall make ample use of the classfication of nondegenerate spaces (E, Φ) with $\dim E = \aleph_0$ when $n = [k : k^2] < \infty$.

Notations. For $\alpha_1, \ldots, \alpha_m \in k$ the space $\langle \alpha_1, \ldots, \alpha_m \rangle$ is spanned by an orthogonal basis $e_1, \ldots, e_m$ with α_i the product of e_i with itself; a denumerably infinite orthogonal sum of lines $\langle \alpha \rangle$ with the same nonzero α throughout is denoted by $E(\alpha)$. P invariably is a hyperbolic plane and $\sum P$ an orthogonal sum of such planes. We have

THEOREM 2 [4]. *Let k be of characteristic 2 and with $[k : k^2] < \infty$, Φ a nondegenerate symmetric bilinear form on the $\aleph_0$-dimensional k vectorspace E. Then*

(i) *E is of the form*:

$$E = E(\gamma_1) \oplus \cdots \oplus E(\gamma_r) \oplus \langle \beta_1\beta_1\beta_2\beta_2 \cdots \beta_s\beta_s\rangle \oplus \langle \alpha_1\alpha_2 \cdots \alpha_t\rangle \quad (r \geqslant 1) \qquad \text{(I)}$$

or

$$E = \sum^{\infty} P \oplus \langle \beta_1\beta_1\beta_2\beta_2 \cdots \beta_p\beta_p\rangle \oplus \langle \alpha_1\alpha_2 \cdots \alpha_q\rangle, \qquad \text{(II)}$$

where all the sums are orthogonal and, in the first case, the elements $\gamma_1,\dots,\gamma_r$, $\beta_1,\dots,\beta_s$, $\alpha_1,\dots,\alpha_t$ *are independent over* k^2 *and the same for* $\beta_1,\dots,\beta_p$, $\alpha_1,\dots,\alpha_q$ *in the second case (thus* $r+s+t \leqslant n$, $p+q \leqslant n$*).*

(ii) *E is uniquely determined, up to orthogonal isomorphism, by its range* $\|E\|$, *the range* $\|E^{\perp\!\!\perp}\|$ *and by the space* $E^{\perp}$. *(In particular, the numbers r, s and t, respectively p and q are orthogonal invariants of the space E.)*

(iii) *In terms of the above bases: If* $\|E^{*\perp\!\!\perp}\| \neq 0$ *(i.e.,* E^* *not closed) then E is of type* (I), *if* $\|E^{*\perp\!\!\perp}\| = 0$ *(i.e.,* E^* *closed) then E is of type* (II). *(Thus* (I) *and* (II) *represent nonisomorphic spaces.) A space of type* (I) *is uniquely determined, up to orthogonal isomorphism, by* $\|E\|$, *the subspace of k (over* k^2*) spanned by the elements* $\gamma_1,\dots,\gamma_r$ *and by the space* $\langle\alpha_1,\dots,\alpha_t\rangle$. *A space of type* (II) *is uniquely determined, up to isomorphism, by* $\|E\|$ *and by the space* $\langle\alpha_1,\dots,\alpha_q\rangle$.

6. Proof of Theorem 1 when $F^{\perp} = (0)$

6.1. *The lattice.* The relevant lattice for case (j) in Theorem 1 is given by the following diagram [5]. Numbers in braces indicate the orthogonals. Indices mentioned are *a*, *b*, *c*, *d*, *g*, *i*. Examples can be given where all 37 elements are different (a field with $[k:k^2] \geqslant 4$ is needed for the purpose).

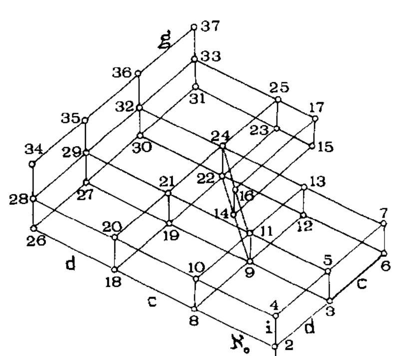

FIG. 1. $\mathcal{T}(F, E^*)$, where $F^{\perp} = (0)$ and $E^* = E^{*\perp\!\!\perp}$ and $[k:k^2] < \infty$, $a+i+d+c = \dim E/E^* \leqslant [k:k^2] < \infty$.

$1 = (0)\ \{37\}$
$2 = F^* \cap E^{*\perp *}\ \{33\}$
$3 = E^{*\perp *}\ \{25\}$
$4 = F \cap E^{*\perp}\ \{31\}$
$5 = E^{*\perp *} + F \cap E^{*\perp}\ \{23\}$
$6 = E^{*\perp} \cap (F \cap E^{*\perp})^{\perp}\ \{17\}$
$7 = E^{*\perp}\ \{15\}$
$8 = F^*\ \{7\}$
$9 = F^* + E^{*\perp *}\ \{7\}$
$10 = F^* + F \cap E^{*\perp}\ \{6\}$
$11 = F^* + E^{*\perp *} + F \cap E^{*\perp}\ \{6\}$
$12 = (F^* + E^{*\perp}) \cap (F \cap E^{*\perp})^{\perp}\ \{5\}$
$13 = F^* + E^{*\perp}\ \{3\}$
$14 = (F + E^{*\perp}) \cap E^*\ \{7\}$
$15 = E^*\ \{7\}$
$16 = (F + E^{*\perp}) \cap E^* + F \cap E^{*\perp}\ \{6\}$
$17 = E^* + F \cap E^{*\perp}\ \{6\}$
$18 = F \cap E^{*\perp * \perp} \cap (F \cap E^{*\perp})^{\perp}\ \{5\}$
$19 = (F + E^{*\perp *}) \cap E^{*\perp * \perp} \cap (F \cap E^{*\perp})^{\perp}\ \{5\}$
$20 = F \cap E^{*\perp * \perp}\ \{3\}$
$21 = (F + E^{*\perp *}) \cap E^{*\perp * \perp}\ \{3\}$
$22 = (F + E^{*\perp}) \cap E^{*\perp * \perp} \cap (F \cap E^{*\perp})^{\perp}\ \{5\}$
$23 = E^{*\perp * \perp} \cap (F \cap E^{*\perp})^{\perp}\ \{5\}$
$24 = (F + E^{*\perp}) \cap E^{*\perp * \perp}\ \{3\}$
$25 = E^{*\perp * \perp}\ \{3\}$
$26 = F \cap (F \cap E^{*\perp})^{\perp}\ \{4\}$
$27 = (F + E^{*\perp *}) \cap (F \cap E^{*\perp})^{\perp}\ \{4\}$
$28 = F \cap (F^* \cap E^{*\perp *})^{\perp}\ \{2\}$
$29 = (F + E^{*\perp *}) \cap (F^* \cap E^{*\perp *})^{\perp}\ \{2\}$
$30 = (F + E^{*\perp}) \cap (F \cap E^{*\perp})^{\perp}\ \{4\}$
$31 = (F \cap E^{*\perp})^{\perp}\ \{4\}$
$32 = (F + E^{*\perp}) \cap (F^* \cap E^{*\perp *})^{\perp}\ \{2\}$
$33 = (F^* \cap E^{*\perp *})^{\perp}\ \{2\}$
$34 = F\ \{1\}$
$35 = F + E^{*\perp *}\ \{1\}$
$36 = F + E^{*\perp}\ \{1\}$
$37 = E\ \{1\}.$

6.2. *A reduction.* Proofs of the type under discussion may often be simplified by chopping off in advance certain finite dimensional orthogonal summands. However, difficulties may arise because the cancellation theorem is not at disposal. We carry out such a preliminary normalization here.

Let $S := F \cap E^{*\perp} = \bar{F} \cap E^{*\perp}$ (space 4 in he diagram); the diagram shows that $R := S^*$ coincides with rad S so $S = R \oplus Q$ for some anisotropic Q with $\dim Q \leqslant \dim S = a + i \leqslant [k : k^2] < \infty$.

Since $F^{\perp} = (0)$ we have that for every $x \in E$ and every finite dimensional $G \subset E$ the manifold $x + G^{\perp}$ meets F.

This fact and the explicit form of E provided by Theorem 2 permit us to find isometric (anisotropic) spaces $R', \bar{R}'$ in $F \cap Q^{\perp}$ and $\bar{F} \cap Q^{\perp}$, respectively, such that we have decompositions as follows. $E = Q \oplus^{\perp} (R \oplus R') \oplus^{\perp} E_0$ with $F = Q \oplus^{\perp} (R \oplus R') \oplus^{\perp} F_0$ and $E^* = R \oplus (E_0)^*$, furthermore $E = Q \oplus^{\perp} (R \oplus \bar{R}') \oplus^{\perp} \bar{E}_0$ with $\bar{F} = Q \oplus^{\perp} (R \oplus \bar{R}') \oplus \bar{F}_0$ and $E^* = R \oplus (\bar{E}_0)^*$. Theorem 2 now tells us again that E_0 and $\bar{E}_0$ are isometric. Since $(E_0)^{*\perp} \cap F_0 = (0) = (\bar{E}_0)^{*\perp} \cap \bar{F}_0$ we have reduced the proof of Theorem 1 to the situation where the space 4 in the lattice is the null space. The lattice then looks as in Fig. 2.

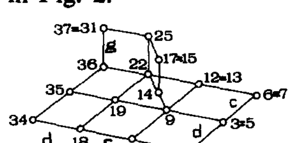

FIGURE 2

There are seven join irreducible elements with three of them, namely, 6, 14, 18 generating nonprime filters.

6.3. *Construction of the initial triple* $(W_0, \bar{W}_0, T_0)$ *(as discussed in* 4.3). By 6.2 we may and shall assume that $\mathscr{V}(F, E^*)$ is given by the diagram of Fig. 2. By Theorem 2 we have a decomposition $E = \sum P \oplus^{\perp} k(x_1, x_1', x_2, x_2', ..., x_d, x_d') \oplus^{\perp} k(y_1, ..., y_c)$ with $\Phi(x_j, x_j) = \Phi(x_j', x_j') = \beta_j$, $\Phi(y_j, y_j) = \alpha_j$ and the scalars $\{\beta_1, ..., \beta_d, \alpha_1, ..., \alpha_c\}$ linearly independent over k^2. Therefore $E^* = \sum P \oplus k(x_1 + x_1', ..., x_d + x_d')$, $E^{*\perp} = k(x_1 + x_1', ..., x_d + x_d') \oplus k(y_1, ..., y_c)$, $E^{*\perp *} = k(x_1 + x_1', ..., x_d + x_d')$, $E^{*\perp*\perp} = \sum P \oplus k(x_1 + x_1', ..., x_d + x_d') \oplus k(y_1, ..., y_c)$.

Since E^* is closed (by assumption) and of finite codimension in E the space $E^* + F$ is both closed and dense, $E^* + F = E$. We decompose $y_j = f_j + e_j$ $(j = 1, ..., c)$, where $f_j \in F$, $e_j \in E^*$ and assert that $\{f_j\}$ is a basis of a supplement of the space 8 in 18 and $\{e_j\}$ a basis of a supplement of 9 in 14. We have at any rate that $f_j \in 18$. Assume that $\sum \lambda_j f_j \in 8$. By considering "lengths" $\|z\| := \phi(z, z)$ we get $0 = \|\sum \lambda_j f_j\| = \sum \lambda_j^2 \|f_j\| = \sum \lambda_j^2 \|y_j\| = \sum \lambda_j^2 \alpha_j$ so $\lambda_1 = \cdots = \lambda_c = 0$ by independence. As $c = \dim 18/8$ the first assertion is proved. The second follows by a similar argument ($E^{*\perp} \cap F = (0)$ is in force).

We also decompose $x_j = g_j + d_j$ $(j = 1, ..., d)$, where $g_j \in F$, $d_j \in E^*$. It follows (as before) that $\{g_j\}$ is a basis of a supplement of 18 in 26.

We may always change the $c + d$ vectors f_j, g_r modulo the space F^* and still have bases of supplements as indicated. We can do this in such a fashion that the modified vectors are pairwise orthogonal. Indeed, if $X := F^* \cap k(f_j, g_r)^{\perp}$ then $\dim X/\mathrm{rad}\, X = \aleph_0$ so that X contains a space Y which is a sum of $c + d$ hyperbolic planes. In Y we find a family f_j^*, g_r^* with products between different members that equal the products between corresponding members of the amily f_j, g_r so that the family $f_j + f_j^*$, $g_r + g_r^*$ is an orthogonal $(c + d)$-tuple. Therefore, we may and shall assume that

$$f_j, g_r \text{ are pairwise orthogonal.} \tag{10}$$

We now define W_0 as the span of the vectors $x_1 + x_1', ..., x_d + x_d'$, $y_1, ..., y_c f_1, ..., f_c$, $g_1, ..., g_d$. $\bar{W}_0$ is defined by an analogously determined family of $2c + 2d$ vectors $\bar{f}_j$, $\bar{g}_r$. $T_0 : W_0 \to \bar{W}_0$ is the linear map which leaves $x_j + x_j'$ and y_j pointwise fixed and sends f_j, g_r into $\bar{f}_j, \bar{g}_r$, respectively. In view of (10) and the corresponding property relating to $\bar{W}_0$ it is very easy to verify that T_0 is an isometry.

6.4. *Verification of the induction assumptions* (6), (7), (8) *in* 4.3. From the construction of $W_0, \bar{W}_0$ we obain $A = A^* + S$, where $S \subset W_0$, and $\bar{A} = \bar{A}^* + T_0(S)$ for all A (Fig. 2). In order to prove (6) it is therefore sufficient to check all $A \subset E^*$. We find $W_0 \cap A = (0)$ for $A = 1, 8$; $W_0 \cap A = k(x_i + x_i')$ when $A = 3, 9$; $W_0 \cap A = k(x_i + x_i', e_j)$ for $A = 14, 17$.

Let us verify the nontrivial inclusion in (7). Since $W_0 + A = W_0 + A^*$ by the construction of W_0 we need only consider all $A, B \subset E^*$. Only $A = 3$, $B = 8$ is not trivial. We have $(W + 3) \cap (W + 8) = W \cap (W + 8) = W = W + (8 \cap 3)$.

6.5. *The assumptions* (5) *and* (9) *of Principle* II *in* 4.3 *are satisfied.* The elements which fall under the jurisdiction of (5) or (9) are $X = 6$, 14, 18, 26. By construction W_0 contains the space 6 and supplements in 14, 18, 26 of 9, 8, 18, respectively.

We may now quote Principle II in 4.3 which terminates the proof of Theorem 1 in this case.

6.6. *Orthogonal sums of minimal pairs.* It is quite simple to construct certain *minimal pairs* (F, E) with $F^\perp = (0)$ in E. $F = E = \langle \beta, \beta \rangle$ or $F = E = \langle \alpha \rangle$ yield pairs with $a = 1$, resp., $i = 1$ and all other quotients defined by their lattices zero. Another example is that of a dense hyperplane in an alternate space E, $E = E^*$; it has $b = \aleph_0$ and $g = 1$ and the other quotients zero. Finally, let F_0 be a dense hyperplane in an alternate space E^*, $E^* = F_0 \oplus^\perp (e)$ and set $E_1 = E^* \oplus^\perp (y)$, $E_2 = E^* \oplus^\perp (x, x')$ with $\|y\| := \phi(y, y)$ and $\|x\| = \|x'\|$ nonzero elements of k. The pair (F_1, E_1) with $F_1 = F_0 \oplus (e + y)$ has $(a, b, c, d, g, i) = (0, \aleph_0, 1, 0, 0, 0)$ and (F_2, E_2) with $F_2 = F_0 \oplus (e + x) \oplus (x')$ has $(a, b, c, d, g, i) = (0, \aleph_0, 0, 1, 0, 0)$.

We can *add pairs* (F_ι, E_ι) by forming the pair (F, E), where F, E are the external orthogonal sums of the F_ι and E_ι, respectively. Given a pair (F, E) with dense F one can form $a + b + c + d + g + i$ minimal pairs of the type described such that their sum has the same lattice as the given pair. Hence from Theorem 1 we obtain

THEOREM 3. *Let F and E be as in* (*j*) *of Theorem* 1. *The pair* (F, E) *is an orthogonal sum of minimal pairs.*

Remark. The representation of (F, E) as a sum of minimal pairs is, of course, *not* unique. This is in the nature of an (F, E) with dense F and *not* a defect of the description. The fact also accounts for the involved proof in Chapter 5; there are no natural choices for representatives in the orbit of F.

7. Proof of Theorem 1 when $F^\perp$ is Trace-Valued

7.1. *The lattice.* In order to avoid excess computation Moresi has distinguished between the following two cases for the computation of $\mathscr{V}(F, E^*)$:

$$\text{Case I:} \qquad F \cap E^{*\perp} \subset E^*,$$

$$\text{Case II:} \qquad F \cap E^{*\perp} \not\subset E^*.$$

THEOREM [5]. *Assume that* $[k:k^2]=1$, $E^* = E^{*\perp\perp}$, $E^{*\perp} \neq E^{*\perp *}$. *Let* $F \subset E$ *be a subspace with* $F^\perp = F^{\perp *}$. *If we are in case I then* $\mathscr{V}(F, E^*)$ *has, in general,* 40 *elements and it is described by Fig.* 3. *If we are in case II then* $\mathscr{V}(F, E^*)$ *has, in general,* 43 *elements and it is given by Fig.* 4.

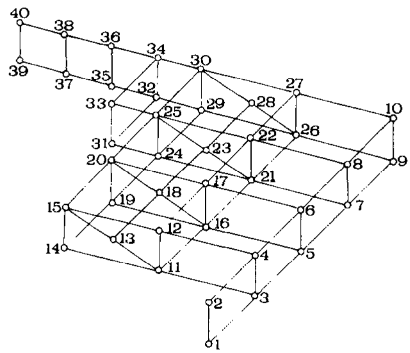

FIGURE 3

$1 = (0)$ $\{40\}$
$2 = E^{*\perp}$ $\{39\}$
$3 = F \cap F^\perp$ $\{38\}$
$4 = E^{*\perp} + F \cap F^\perp$ $\{37\}$
$5 = (F \cap F^\perp)^{\perp\perp}$ $\{38\}$
$6 = E^{*\perp} + (F \cap F^\perp)^{\perp\perp}$ $\{37\}$
$7 = F^{\perp\perp} \cap F^\perp$ $\{36\}$
$8 = E^{*\perp} + F^{\perp\perp} \cap F^\perp$ $\{35\}$
$9 = F^\perp$ $\{33\}$
$10 = F^{*\perp}$ $\{31\}$
$11 = F^*$ $\{10\}$
$12 = F^* + E^{*\perp}$ $\{9\}$
$13 = F$ $\{9\}$
$14 = (F + E^{*\perp})^*$ $\{10\}$
$15 = F + E^{*\perp}$ $\{9\}$
$16 = F^* + (F \cap F^\perp)^{\perp\perp}$ $\{10\}$
$17 = F^* + E^{*\perp} + (F \cap F^\perp)^{\perp\perp}$ $\{9\}$
$18 = F + (F \cap F^\perp)^{\perp\perp}$ $\{9\}$
$19 = (F + E^{*\perp} + (F \cap F^\perp)^{\perp\perp})^*$ $\{10\}$
$20 = F + E^{*\perp} + (F \cap F^\perp)^{\perp\perp}$ $\{9\}$
$21 = F^* + F^{\perp\perp} \cap F^\perp$ $\{10\}$
$22 = F^* + E^{*\perp} + F^{\perp\perp} \cap F^\perp$ $\{9\}$
$23 = F + F^{\perp\perp} \cap F^\perp$ $\{9\}$
$24 = (F + E^{*\perp} + F^{\perp\perp} \cap F^\perp)^*$ $\{10\}$
$25 = F + E^{*\perp} + F^{\perp\perp} \cap F^\perp$ $\{9\}$
$26 = F^* + F^\perp$ $\{8\}$
$27 = F^* + F^{*\perp}$ $\{7\}$
$28 = F + F^\perp$ $\{7\}$
$29 = (F + F^{*\perp})^*$ $\{8\}$
$30 = F + F^{*\perp}$ $\{7\}$
$31 = F^{*\perp\perp}$ $\{10\}$
$32 = F^{*\perp\perp} + F^\perp$ $\{8\}$
$33 = F^{\perp\perp}$ $\{9\}$
$34 = F^{\perp\perp} + F^\perp$ $\{7\}$
$35 = (F^{*\perp\perp} + F^\perp)^{\perp\perp}$ $\{8\}$
$36 = (F + F^\perp)^{\perp\perp}$ $\{7\}$
$37 = (F \cap F^\perp)^{\perp *}$ $\{6\}$
$38 = (F \cap F^\perp)^\perp$ $\{5\}$
$39 = E^*$ $\{2\}$
$40 = E$ $\{1\}$.

Remark. Examples can be given (over any field) where all fourty elements in Fig. 3 are different.

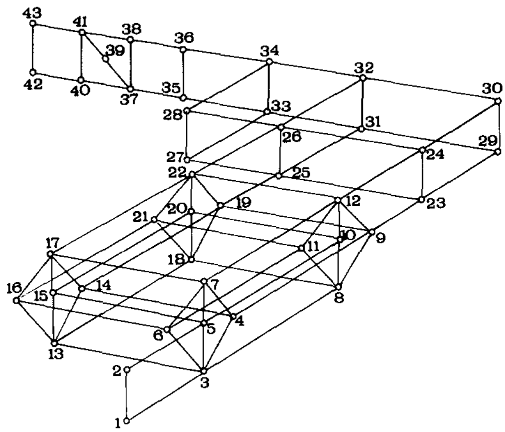

FIGURE 4

$1 = (0)$ {43}
$2 = E^{*\perp}$ {42}
$3 = F \cap F^\perp$ {41}
$4 = (F + E^{*\perp}) \cap F^\perp$ {38}
$5 = F \cap F^\perp + E^{*\perp}$ {40}
$6 = F \cap F^{*\perp}$ {39}
$7 = (F + E^{*\perp}) \cap F^{*\perp}$ {37}
$8 = (F \cap F^\perp)^{\perp\!\!\perp}$ {41}
$9 = [(F \cap F^\perp)^{\perp\!\!\perp} + F + E^{*\perp}] \cap F^\perp$ {38}
$10 = (F \cap F^\perp)^{\perp\!\!\perp} + E^{*\perp}$ {40}
$11 = (F \cap F^\perp)^{\perp\!\!\perp} + (F \cap F^{*\perp})$ {39}
$12 = [(F \cap F^\perp)^{\perp\!\!\perp} + F + E^{*\perp}] \cap F^{*\perp}$ {37}
$13 = F^*$ {30}
$14 = (F + E^{*\perp})^*$ {30}
$15 = F^* + E^{*\perp}$ {29}
$16 = F$ {29}
$17 = F + E^{*\perp}$ {29}
$18 = F^* + (F \cap F^\perp)^{\perp\!\!\perp}$ {30}
$19 = (F + (F \cap F^\perp)^{\perp\!\!\perp} + E^{*\perp})^*$ {30}
$20 = F^* + (F \cap F^\perp)^{\perp\!\!\perp} + E^{*\perp}$ {29}
$21 = F + (F \cap F^\perp)^{\perp\!\!\perp}$ {29}
$22 = F + (F \cap F^\perp)^{\perp\!\!\perp} + E^{*\perp}$ {29}
$23 = F^{\perp\!\!\perp} \cap F^\perp$ {36}
$24 = F^{\perp\!\!\perp} \cap F^{*\perp}$ {36}
$25 = F^* + F^{\perp\!\!\perp} \cap F^\perp$ {30}
$26 = F + F^{\perp\!\!\perp} \cap F^\perp$ {29}
$27 = F^{*\perp\!\!\perp}$ {30}
$28 = F^{\perp\!\!\perp}$ {29}
$29 = F^\perp$ {28}
$30 = F^{*\perp}$ {27}
$31 = F^* + F^\perp$ {24}
$32 = F + F^\perp$ {23}
$33 = F^{*\perp\!\!\perp} + F^\perp$ {24}
$34 = F^{\perp\!\!\perp} + F^\perp$ {23}
$35 = (F^* + F^\perp)^{\perp\!\!\perp}$ {24}
$36 = (F + F^\perp)^{\perp\!\!\perp}$ {23}
$37 = [(F + E^{*\perp}) \cap F^{*\perp}]^\perp$ {12}
$38 = [(F + E^{*\perp}) \cap F^\perp]^\perp$ {9}
$39 = (F \cap F^{*\perp})^\perp$ {11}
$40 = (F \cap F^\perp)^{\perp *}$ {10}
$41 = (F \cap F^\perp)^\perp$ {8}
$42 = E^*$ {2}
$43 = E$ {1}.

Remark. Examples can be given (over any field) where all 43 elements in Fig. 4 are different. Notice that $\dim 7/6 = \dim 6/3 = \dim 5/3 = 1$; all specializations are nondistributive.

7.2. *The construction of the initial triple* $(W_0, \bar{W}_0, T_0)$ *in case I.* The join-irreducible lattice elements which are not trace-valued or which do not generate prime filters are $D = 2$, 13, 14 (see Fig. 3). We decompose $E = E^* \oplus^\perp (g)$ (where g spans $E^{*\perp}$), $F = F^* \oplus (p+g)$, $\bar{F} = \bar{F}^* \oplus (\bar{p}+g)$, where p, $\bar{p} \in E^*$. We set $W_0 = k(g, p+g)$, $\bar{W}_0 = k(g, \bar{p}+g)$. The assignement $g \to g$, $p+g \to \bar{p}+g$ defines an isometry $T_0 : W_0 \to \bar{W}_0$. (g) is a supplement of space 1 in 2, $(p+g)$ is a supplement of 11 in 13 and (p) is a supplement of 11 in 14 since $14 = (F + E^{*\perp})^* = (F^* + (p+g) + (g))^* = F^* + (p) = 11 + (p)$.

Thus conditions (9) and (5) of 4.3 are satisfied.

As to condition (6) of 4.3 we remark the following. If $A \subset E^*$ (hence $\bar{A} \subset E^*$ as well) then $A \cap W_0 = (0)$ if $A \not\supset 14$ and $A \cap W_0 = (p)$ if $A \supset 14$; therefore $T_0(W_0 \cap A) = \bar{W}_0 \cap \bar{A}$ in either case. If, on the other hand, $A \not\subset E^*$ then $A \cap W_0 = (g)$ when $A \not\supset 13$ and $A \not\supset 14$, $A \cap W_0 = (p+g)$ when $A \supset 13$ and $A \not\supset 14$ and finally $A \cap W_0 = W_0$ when $A \supset 14$. We see that condition (6) of 4.3 is satisfied.

7.3. *The verification of conditions* (7) *and* (8) *in* 4.3. To verify these conditions by cases is exceedingly cumbersome. The following argument, which once more makes full use of the diagram, is very neat. Since W_0 contains supplements of 1 in 2 and of 11 in 12, 13, 14 we have $W_0 + A = W_0 + A^*$ for all $A \in \mathscr{V}(F, E^*)$. Since furthermore the sublattice $[(0), E^*]$ of $\mathscr{V}(F, E^*)$ is distributive, we shall easily see that $(W_0 + A^*) \cap (W_0 + B^*) = W_0 + A^* \cap B^*$ which provides us with the desired inclusion $(W_0 + A) \cap (W_0 + B) \subset W_0 + (A \cap B)$. Indeed, if $v := \lambda p + \mu g + a = \lambda' p + \mu' g + b$ is an element of the intersection $(W + A^*) \cap (W + B^*)$ then a check on $\Phi(v, v)$ gives $\mu = \mu'$. Thus $(\lambda - \lambda') p \in A^* + B^*$. Now 14 generates in $[(0), E^*]$ a prime filter (being a join irreducible element in a distributive lattice). Therefore either $\lambda - \lambda' = 0$ and so $a = b \in A^* \cap B^*$ or else $\lambda - \lambda' \neq 0$ and $p \in A^*$ or $p \in B^*$. In either case $v \in W_0 + (A^* \cap B^*)$. Q.E.D.

7.4. *Remark.* There is an alternative for proving case I. Let $\mathscr{W}$, $\bar{\mathscr{W}}$ be the lattices generated orthostably by F^* and $\bar{F}^*$, respectively, in the nondegenerate alternate space E^*. $p \to \bar{p}$ induces an isometry between the lines $W_0 := (p)$, $\bar{W}_0 := (\bar{p})$. One can show that $(W_0, \bar{W}_0, T_0)$ satisfies the assumptions of Principle II with $E^*, \mathscr{W}$ in the role of $E, \mathscr{V}$ so that we obtain an isometry $T^* : E^* \to E^*$ with $T^*(F^*) = \bar{F}^*$ and $T^* p = \bar{p}$. T^* can be extended (to an isometry) by identity on $E^{*\perp} = (g)$.

7.5. *The construction of the initial triple* $(W_0, \bar{W}_0, T_0)$ *in case* II. The join irreducible lattice elements which are not trace-valued or which do not generate prime filters are $D = 2, 4, 6, 39, 40$. Let again $E = E^* \oplus^\perp (g)$; g spans $E^{*\perp}$. Since we are in case II there is a vector $x \in 6 \backslash E^*$,

$x = p + g (p \in E^*)$. $(p+g)$ is a supplement of 3 in 6. (p) must be a supplement of 3 in 4 since $4 = 7 \cap E^* = (6+2)^* = (3 + k(p+g, g))^* = 3 + k(p+g, g)^* = 3 + (p)$. Furthermore there is a vector $y = p' + g \in 39 \backslash E^* (p' \in E^*)$. $(p' + g)$ is a supplement of 37 $(= 39^*)$ in 39. Now we see that (p') is a supplement of 37 in 40 since $40 = 41 \cap E^* = (39+2)^* = (37 + (p'+g) + (g))^* = 37 + (p')$. Because $p' + g \in 39 = 6^\perp$ we get $p' + g \perp p + g$ so $\Phi(p, p') = \Phi(g, g)$. We set

$$W_0 := (g) \oplus k(p, p'), \qquad \text{where} \quad \Phi(g, g) = \Phi(p, p').$$

$\bar{W}_0 := (g) \oplus k(\bar{p}, \bar{p}')$ with $\Phi(\bar{p}, \bar{p}') = \Phi(g, g)$ is determined analogously. $T_0 : W_0 \to \bar{W}_0$ is the isometry which sends g in g, p in $\bar{p}$, p' in $\bar{p}'$.

The verification of condition (6) in 4.3 is routine: $W_0 \cap A$ equals (0) when $A \in [1, 18]$; $W_0 \cap A$ equals (g) when $A \in [2, 20]$; $W_0 \cap A = (p)$ if $A \in [4, 37]$; $W \cap A$ is $(p+g)$ when $A \in [6, 21]$; $W_0 \cap A$ is the plane $k(p, g)$ when $A \in [7, 38]$; $W_0 \cap A = k(p, p'+g)$ for $A = 39$ and $W_0 \cap A = k(p, p')$ when $A = 40, 42$; finally $W_0 \cap A$ is all of W_0 when $A = 41, 43$.

The verification of (7) and (8) on the other hand can be done just as in 7.3. We may therefore quote Principle II of 4.3 and have proved the theorem in case II as well.

Certainly a remark analogous to 7.4 can be made here as well.

8. Remarks on the Proof of Theorem 1 when F Is Trace-Valued

In [5] we have the following

Theorem. *Assume that E is nondegenerate and $F \subset E^* = E^{*\perp\!\!\perp}$ and $[k : k^2] < \infty$. Then $\mathscr{V}(F, E^*)$ has, in general, 55 elements and is distributive.*

There are at most five join irreducible elements X with $X \neq X^*$ in the lattice. If k is assumed perfect then only three can be different among them. An initial triple $(W_0, \bar{W}_0, T_0)$ which qualifies for Principle I of 4.3 can be determined along the lines illustrated in the two previous chapters.

9. Remarks on the Proof of Theorem 1 when F is Totally Isotropic

To know the orbits of totally isotropic subspaces is of great practical value for the actual computational work with spaces (Witt decompositions). First of all we have

Theorem [5]. *Assume that E is nondegenerate and $F \subset F^\perp \subset E$ and $[k : k^2] < \infty$. $\mathscr{V}(F, E^*)$ has, in general, 43 elements and is distributive.*

This is a remarkable example of a *finite* lattice $\mathscr{V}(F, E^*)$, where we have no assumptions on E^* (cf. Chap. 11).

The proof of Theorem 1 in the present case quickly reduces to the situation where $E^{*\perp} = E^{*\perp *}$ and $F \cap E^{*\perp} = (0)$ by chopping off suitable (finite dimensional) orthogonal summands. In this situation $\mathscr{V}(F, E^*)$ has at most 25 elements with not more than five irreducible elements $X \not\subset E^*$.

We remark that without restrictions on k or E^* as expressed in Theorem 1 the conclusion of the theorem will no hold since the lattice cannot rule over such arithmetical invariants as $\|X\|$, where, say, $X = E^{*\perp\perp} \cap F^{\perp *\perp}$ (which has the alternate space $X_0 = F^{\perp *\perp *}$ as its immediate antecedent).

However, if k is perfect, then the proof culminating in an application of Principle I can be carried out. The same holds if k is not perfect but $E^{*\perp\perp} = E^*$ (the initial tripel $(W_0, \bar{W}_0, T_0)$ will then have to consist of isometric supplements of $F^{\perp *}$ in $F^\perp$ and $\bar{F}^{\perp *}$ in $\bar{F}^\perp$, respectively.

(It follows from Theorem 2 that for totally isotropic F with $F \cap E^{*\perp} = (0)$, $F^\perp$ contains a supplement W_0 of $F^{\perp *}$ isometric to $\langle \beta_1, \beta_1, \ldots, \beta_s, \beta_s \rangle$.)

10. A Theorem on Witt Decompositions

From (jv), Theorem 1 one can deduce the following useful information on Witt decompositions

THEOREM 4. *Let E be nondegenerate and of dimension at most $\aleph_0$ and with $E^* = E^{*\perp\perp}$. Assume that $[k : k^2] < \infty$. In order that a totally isotropic subspace $F \subset E$ admit a Witt decomposition, $E = (F \oplus F') \oplus^\perp E_0$ for some totally isotropic F', it is necessary and sufficient that $F^{\perp\perp} = F$ and $F \cap E^{*\perp} = (0)$.*

11. Remarks on the Proof of Theorem 1 when F Is Trace-Valued and E^* Dense

It seems considerably harder to produce *finite* lattices $\mathscr{V}(F, E^*)$ when E^* is not closed, even when E^* is a hyperplane. Here is an example with the additional feature of being not distributive. It is an amusing task to construct examples where all the conditions involved are satisfied.

THEOREM [5]. *Let E be a nondegenerate space with $E^{*\perp} = (0)$ and E/E^* of dimension* 1. *Let F be a subspace with* (i) $\dim F/F^* = 1$, (ii) $F^{*\perp\perp}/F^{*\perp\perp *} = 1$, (iii) $\dim F^{\perp\perp}/F^{*\perp\perp} = 1$, (iv) $F^{*\perp}/F^{*\perp *} = 0$,

(v) $(F^{\perp\!\!\perp} \cap F^{*\perp}) = F^{*\perp\!\!\perp} \cap F^{\perp}$, (vi) $(F \cap F^{\perp})^{\perp * \perp\!\!\perp} = (F \cap F^{\perp})^{\perp}$. *Then* $\mathscr{V}(F, E^*)$ *is a nondistributive lattice with at most* 37 *elements.*

A proof for Theorem 1 consists again in the construction of $(W_0, \bar{W}_0, T_0)$ which qualifies for an application of Principle II. (F and $F^{*\perp\!\!\perp}$ are not trace-valued and $F^{\perp\!\!\perp *}$ is trace-valued but does not generate a prime filter; all the other critical elements are join reducible. W_0 is $k(x, y)$ with $F^* \oplus (x) = F$ and $F^{*\perp\!\!\perp *} \oplus (y) = F^{*\perp\!\!\perp}$. One can arrange for $x \perp y$ and $\Phi(x, x) = \Phi(y, y)$ so that $x + y$ spans a supplement of $F^{*\perp\!\!\perp *}$ in $F^{\perp\!\!\perp *}$. Verifications run just as in 7.3.)

12. The Finite Dimensional Case

Assume k perfect so $\dim E/E^* \leqslant 1$. The product on E is assumed to be nondegenerate. We give here the generalization of Witt's theorem; it illustrates the natural role of certain lattices. The verifications of the statements in this short chapter are left to the reader.

For F a linear subspace of E we call *reducible* the pair (F, E) if E is an orthogonal sum, $E = E_1 \oplus^{\perp} E_2$ such that $E^* = E_1^* \oplus E_2^*$ and $F = (F \cap E_1) \oplus (F \cap E_2)$. What are (modulo metric automorphisms of E) the irreducible pairs (F, E) when E is finite dimensional and nondegenerate? Clearly, when $E = E^*$ ("trace-valued case") then E is a hyperbolic plane and $\dim F$ is either 0 or 1 or 2. If $E \neq E^*$, or equivalently, $E^{*\perp} \neq (0)$, then we distinguish two cases.

Case A. $E^{*\perp} \not\subset E^*$. Hence $E = E^* \oplus E^{*\perp}$. There are four irreducible pairs. E is a nonisotropic line and F is either (0) or E; E is then orthogonal sum $E = (e_1, e_2) \oplus (l)$ of a hyperbolic plane and a nonisotropic line (l) and F is either the line $(e_2 + l)$ or the plane $(e_1) \oplus (e_2 + l)$.

Case B. $E^{*\perp} \subset E^*$. Here we get five irreducible pairs. E is a plane spanned by a basis (l, e) for which the matrix of the form is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and F is either (0) or E or one of the lines (l), (e); the fifth pair is given by E, where E is an orthogonal sum of a hyperbolic plane P with symplectic basis (e_1, e_2) and the plane (l, e); F is the plane $(e_1) \oplus (e_2 + l)$.

Theorem 5. *Let the field k be perfect. Every pair (F, E), where E is nondegenerate and finite dimensional, is an orthogonal sum of irreducible pairs; the decomposition into irreducible pairs is unique up to isometry. If $E^{*\perp} = (0)$ then there are (always up to isometry) 3 irreducible pairs; if $E^{*\perp} \not\subset E^*$ then there are 4 irreducible pairs; if $E^{*\perp} \cap E^* \neq (0)$ then there are 5 irreducible pairs. An arbitrary pair (F, E) is characterized modulo metric automorphisms of E be the dimension of the spaces E/E^*, $E^*/r(E^*)$, $E^{*\perp} \cap F$, $F^*/r(F^*)$, $r(F^*)/r(F)$ (here we have set $r(X) := X \cap X^{\perp}$).*

COROLLARY ("Cancellation Theorem"). *Let k be perfect, E of finite dimension and equipped with a nondegenerate symmetric bilinear form (which is not necessarily trace-valued). Let $F, \bar{F}$ be subspaces of E. In order that there exist an isometry $T: E \to E$ with $T(F) = \bar{F}$ it is necessary and sufficient that there is a lattice isomorphism $\tau: \mathscr{V}(F, E^*) \cong \mathscr{V}(\bar{F}, E^*)$ which respects $*$ and $^\perp$ and indices and which sends F into $\bar{F}$.*

A careful inspection of cases shows that the possibility of extending a *given* isometry $T_0: F \to \bar{F}$ is ruled by the lattice as well:

THEOREM 6. *Let the space E be as in the Corollary of Theorem* 1 *and $T_0: F \to \bar{F}$ an isometry between subspaces $F, \bar{F}$ of E. In order that T_0 admits an extension T (isometry) to all of E it is necessary and sufficient that the following two conditions are satisfied.* (i) *There is at least one lattice isomorphism $\tau: \mathscr{V}(F, E^*) \cong \mathscr{V}(\bar{F}, E^*)$ that respects $*$ and $^\perp$ and has $F^\tau = \bar{F}$.* (ii) *T_0 leaves the space $E^{*\perp} \cap E^* \cap F$ pointwise fixed.*

Remarks. (1) Condition (ii) in Theorem 6 is automatically satisfied when $F \not\subset E^*$.

(2) In contrast to Theorem 5 we have no need to assume in Theorem 6 that τ respects indices: necessary is merely that the assignment $F \to \bar{F}$ can be extended to the $\perp$- and $*$-stable lattices of F and $\bar{F}$. The standard "counter examples to Witt's theorem" such as $\langle \alpha, \alpha \rangle \oplus \langle \alpha \rangle = P \oplus \langle \alpha \rangle$ simply violate this obvious requirement.

(3) Theorem 6 quickly reduces to the well known statement of Witt's theorem in the classic situtíon. For, condition (ii) is trivial when $E = E^*$ and $\mathscr{V}(F, E^*)$ reduces to $\{(0), r(F), F, F^\perp, F + F^\perp, E\}$; by the assumption on T_0 all of the indices equal the corresponding indices in $\mathscr{V}(\bar{F}, E^*)$ and (i) is satisfied.

Note added in proof. Articles which have appeared after this paper was submitted and which contain major contributions to our topic are: P. AMPORT, Teilraumverbände in überabzählbardimensionalen Sesquilinearräumen, Ph. D. Thesis Univ. of Zürich 1978. W. BÄNI, Application of the lattice method to infinite dimensional hermitian spaces, Habilitationsschrift, Univ. of Zürich, 1981. H. GROSS, The lattice method in the theory of quadratic spaces of non-denumerable dimensions, *J. Algebra*, in press. H. GROSS, "Quadratic Forms in Infinite Dimensional Vector Spaces," Progress in Mathematics, Vol. 1 Birkhäuser-Verlag, Boston (1979). H. GROSS AND H. A. KELLER, On the problem of classifying infinite chains in projective and orthogonal geometry, to appear. L. HAAPASALO, Von Vektorraumisometrien induzierte Verbandsisomorphismen zwischen nicht orthostabilen und nicht distributiven Vektorraumverbänden, *Ann. Acad. Sci. Fenn. Ser. A.*, in press. R. MORESI, Untersuchungen in abzählbardimensionalen nicht spurwertigen ε-hermitischen Räumen, Ph. D. Thesis, Univ. of Zürich 1980. M. SAARIMÄKI, Zur Klassifikation von Paaren dichter Teilräume in hermitischen Räumen von abzählbarer Dimension, *Ann. Acad. Sci. Fenn. Ser. A, Dissertationes* **34**, 1981.

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